4.1 Vector Spaces and Subspaces

The set $\mathbb{R}^n$, where $n \geq 1$, is an example of the vector spaces. In general, we have:

Definition (Vector Space)

A **vector space** is a non-empty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Examples of Vector Spaces

1. The spaces $\mathbb{R}^n$, where $n \geq 1$.

   Note $\mathbb{R}^n$ satisfies all the 10 rules above.

2. For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$\mathbf{p}(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n \tag{1}$$

where the coefficients $a_0,\ldots,a_n$ and the variable t are real numbers. The degree of $\mathbf{p}$ is the highest power of t in (1) whose coefficient is not zero. If $\mathbf{p}(t)=a_0\neq 0$, the degree of $\mathbf{p}$ is zero. If all the coefficients are zero, $\mathbf{p}$ is called the *zero polynomial*. We can check $\mathbb{P}_n$ satisfies all the 10 rules.
For example, if $q(t)=b_0+b_1t+b_2t^2+\cdots+b_nt^n$. The the sum of $p(t)$ and $q(t)$ is in $\mathbb{P}_n$ (rule 1) In fact, $(p+q)(t)=p(t)+q(t)=(a_0+b_0)+(a_1+b_1)t+\cdots+(a_n+b_n)t^n$.
is also a polynomial of degree n.

3. Let V be the set of all real-valued functions defined on a set $\mathbb{D}$. (Typically, $\mathbb{D}$ is the set of real numbers or some interval on the real line.)

   V satisfies all the 10 properties.

   For example: Let $\mathbb{D}=\mathbb{R}$, $f(t)=2t+e^t$, $g(t)=\cos t$. $\in V$.

   And $f(t)+g(t)$, $3f(t)$ are also in V.

Example 1. Let V be the first quadrant in the xy-plane; that is, let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is not in V. (This is enough to show that V is not a vector space.)

ANS: a) Yes. If $\vec{u}$ and $\vec{v}$ are in V, then their entries are non-negative. Since the sum of non-negative numbers are non-negative, the vector $\vec{u}+\vec{v}$ has non-negative entries. Thus $\vec{u}+\vec{v}$ is in V.

b). $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $c = -1$, Then $\vec{u}$ is in V but $c\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ is not in V. So V does not satisfy rule 6 in the definition of vector space. Thus V is not a vector space.

Subspaces

Recall in Section 2.8, we talked about the subspaces of $\mathbb{R}^n$. This notion can be generalized to the subspace of a vector space.

Definition (Subspace).

A **subspace** of a vector space V is a subset H of V that has three properties:
a. The zero vector of V is in H.
b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Example 2. Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

1. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

No. The zero vector in the space $\mathbb{P}_n$ is the constant 0. Notice that the zero vector is not in the given set, so it is not a subspace.

2. All polynomials of degree at most 3, with integers as coefficients.

No, since the set is not closed under scalar multiplication. E.g. $p(t) = t^3$ is in the set but $\frac{1}{2}p(t) = \frac{1}{2}t^3$ is not in the set.

A Subspace Spanned by a Set

Example 3 Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

ANS: We show H satisfies all the three properties in the definition of subspace:

1) $\vec{0} \in H$ since $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$

2) H is closed under addition. In fact, let $\vec{u}, \vec{v} \in H$, then $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2$ and $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
Then $\vec{u}+\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + b_1\vec{v}_1 + b_2\vec{v}_2 = (a_1+a_2)\vec{v}_1 + (b_1+b_2)\vec{v}_2$, which is in H.

3). H is closed under scalar multiplication. Since if c is a scalar, then $c\vec{u} = c(a_1\vec{v}_1 + a_2\vec{v}_2) = ca_1\vec{v}_1 + ca_2\vec{v}_2$, which is still in H.

Theorem 1.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

We call Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 4. Let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

1. $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$

Since $\vec{0}$ is not in W, W is not a vector space.

2. $\begin{bmatrix} \overset{0}{\overset{\shortparallel}{-a+1}} \\ \underset{=0}{a-6b} \\ \underline{2b+a} \end{bmatrix} \begin{matrix} \Rightarrow \underline{a=1} \\ \Rightarrow \underline{b=\frac{1}{6}} \\ \neq 0 \end{matrix}$

Since $\vec{0}$ is not in W. W is not a vector space.

3. $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

Let $\vec{w} \in W$.

$$\vec{w} = \begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a set spans W.

So W is a vector space by Thm 1.

Exercise 5. Let W be the set of all vectors of the form $\begin{bmatrix} 3b+4c \\ c \\ b \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

ANS: Note $\begin{bmatrix} 3b+4c \\ c \\ b \end{bmatrix} = b\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$.

So $W = \text{span}\{\vec{u}, \vec{v}\}$, where $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$.

W is a subspace of $\mathbb{R}^3$ by Thm 1.

Exercise 6. Determine if the set H of all matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

Remark: The set $M_{m\times n}$ of all $m\times n$ matrices is a vector space, under the usual operations of addition of matrices and scalar multiplication.

ANS: Yes, H is a subspace of $M_{2\times 2}$. Since ① zero matrix is in H (by setting $a=b=c=0$). ② the sum of two lower triangular matrices is lower triangular, and ③ the scalar multiplication of a lower triangular is still lower triangular.

Exercise 7. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times 3}$, the vector space of 3×3 matrices.

ANS: S is a subspace of $M_{3\times 3}$ since:

① The zero matrix is in S, as the transpose of a zero matrix is still a zero matrix

② S is closed under addition. Since for any A, B in S, then $A^T = A$ and $B^T = B$. Their sum $A+B$ satisfies $(A+B)^T = A^T + B^T = A+B$.

③ S is closed under scalar multiplication. Since if $A \in S$, then $(cA)^T = cA^T = cA$, i.e. $cA \in S$.